A DIRECT APPROACH TO THE DERIVATION OF ELECTRIC DYADIC GREEN'S FUNCTIONS

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Summary

A straightforward yet rigorous approach that does not require distribution theory is used to derive a generalized electric dyadic Green's function which remains valid within the source region. Although the electric field expressed by the dyadic Green's function proves to be unique, the exact form of the dyadic itself depends upon the shape of its "principle volume." The dependence on principle volume is determined explicitly, and the different Green's dyadics derived by previous authors are shown to emerge merely through the appropriate choice for the principle volumes.

Specifically, we can begin with the vector Helmholtz equation in free space (a similar analysis applies to bounded regions as well) with equivalent current source $\bar{J}_e$ contained within the volume V_J:

$$\nabla^2 \left(\bar{E} - \frac{\bar{J}_e}{i\omega\varepsilon_o} \right) + k^2 \left(\bar{E} - \frac{\bar{J}_e}{i\omega\varepsilon_o} \right) = \frac{\nabla\times\nabla\times\bar{J}_e}{i\omega\varepsilon_o} . \tag{1}$$

For sufficiently well-behaved $\bar{J}_e$, (1) has the solution

$$\bar{E}(\bar{r}) - \frac{\bar{J}_e}{i\omega\varepsilon_o} = - \frac{1}{4\pi i\omega\varepsilon_o} \lim_{V'_\varepsilon \to 0} \int_{V'_J - V'_\varepsilon} \frac{\nabla'\times\nabla'\times\bar{J}_e}{|\bar{r}'-\bar{r}|} e^{ik|\bar{r}'-\bar{r}|} dV', \tag{2}$$

where V'_ε is a small "principle volume" which encloses the singularity at $\bar{r}' = \bar{r}$.

Although the value of the integral in (2) converges independently of the shape of the principle volume V'_ε for $V'_\varepsilon \to 0$,[1] in source regions the limit must be retained explicitly when (2) is differentiated through the integral sign or transformed by integral formulas. In fact, if (2) is substituted directly into (1) and the differentiation brought carelessly under the integral sign, as is done in many textbooks, we find that (1) is not satisfied in the source regions. (Assuming a delta function at $\bar{r}' = \bar{r}$ for $(\nabla^2+k^2)\dfrac{e^{ik|\bar{r}'-\bar{r}|}}{|\bar{r}'-\bar{r}|}$ does not help since the singular point $\bar{r}' = \bar{r}$ is excluded in (2) by V'_ε.) However, if we realize that V'_ε represents a limit of integration which is a function of $\bar{r}$, the variable of differentiation, it can be proven rigorously by direct substitution that (2) is a solution to (1). Similarly, it is a simple matter to show that the familiar vector and dyadic integral formulas[2] apply correctly only if the singularity at $\bar{r}' = \bar{r}$ is enclosed by the principle volume.

With proper treatment of the arbitrarily shaped (regular, of course) principle volume, equation (2) can be recast into the form similar to Van Bladel's[3] result for the spherical principle volume,

$$\bar{E}(\bar{r}) = -\frac{1}{4\pi i\omega\varepsilon_o}\left[\lim_{V'_\varepsilon\to 0}\int_{V'_J-V'_\varepsilon}\bar{J}_e\cdot\bar{\bar{G}}_{eo}\,dV' + \bar{J}_e(\bar{r})\cdot\bar{\bar{L}}\right] \tag{3}$$

where

$$\bar{\bar{G}}_{eo}(\bar{r},\bar{r}') = \nabla(\nabla\psi) + k^2\psi\bar{\bar{I}}$$

$$(\psi = e^{ikR}/R,\ R = |\bar{r}'-\bar{r}|) \tag{4a}$$

is the conventional electric dyadic Green's function (in this example, for free space) valid outside the source region, and

$$\bar{\bar{L}} = -\lim_{S_\varepsilon\to 0}\oint_{S_\varepsilon}\frac{\hat{e}_R\hat{n}}{R^2}\,da \tag{4b}$$

is a dyadic whose value depends on the shape of the principle volume V_ε and which contributes to $\bar{E}$ only at the source points.

The value of each term separately depends upon the shape of the principle volume, but the combined sum does not. In addition, (3) demonstrates clearly that a single dyadic is not sufficient to represent the solution to $\bar{E}$ in terms of $\bar{J}_e$. However, we can write symbolically the complete Green's dyadic in the shorthand form previous authors[3,4,5] have used,

$$\bar{\bar{G}}_e = \bar{\bar{G}}_{eo} + \bar{\bar{L}}\delta(\bar{r}'-\bar{r}), \tag{5}$$

provided it is understood that (5) has meaning only in the context of the two terms specified in (3).

Finally, $\bar{L}$ has been evaluated from (4b) for a number of principle volume shapes. It is found that for a spherical principle volume the Green's dyadic reduces to that of Wilcox[6] and Van Bladel;[3] for a flattened right cylinder with axis along the z-direction, the Green's dyadics of Tai,[4] Collin,[7] and Rahmat-Samii[5] emerge.

In short, discrepancies between the electric dyadic Green's functions derived by different authors in the past are shown to be explainable and reconcilable merely through the proper choice of the principle volume. Moreover, distribution theory alone is not adequate to extract uniquely the proper electric dyadic Green's function in the source region.

References

1. For a rigorous proof that (2) is a solution to (1) which converges independently of the shape of its principle volume V_ε', see theorem 7 in Claus Müller's book, Foundations of the Mathematical Theory of Electromagnetic Waves (Springer-Verlag, New York, 1969).

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